PROPOSITION IN TRANSVERSALS.

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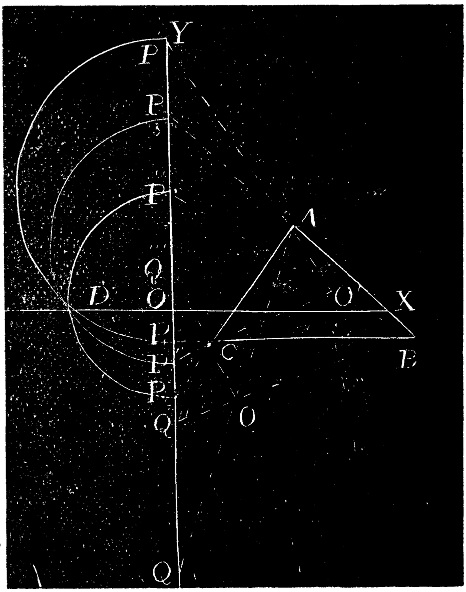
LET any transversal OY be drawn to the triangle ABC, cutting its sides in P_1, P_2 and P_3: from any point, O, on the transversal, lay off distances OP', OP'' and OP''' such that $OP_1 \times OP' = OP_2 \times OP'' = OP_3 \times OP''' = \overline{OD}^2 = c^2$, say: also lay off distances OQ', &c., in the opposite direction from O, such that $OP_1 \times OQ' = OP_2 \times OQ'' = OP_3 \times OQ''' = c^2$: then $P'A$, $P''B$ and $P'''C$ will meet in a point O', and $Q'A$, $Q''B$ and $Q'''C$ will also meet in a point, O''.



The distance OD may be of any length.

Through O, draw OX perpendidular to OY and take these lines for the axes of reference. The equations of BC, CA and AB will be

$$y - m_1x - b_1\ , \qquad (1)$$

$$y - m_2x - b_2\ , \qquad (2)$$

$$y = m_3x + b_3, \qquad (3)$$

so that $b_1 = OP_1$, $b_2 = OP_2$ and $b_3 = OP_3$. The equations of lines through A, B and C will be

$$y - m_2x - b_2 = k_1(y - m_3x - b_3), \qquad (4)$$

$$y - m_3x - b_3 = k_2(y - m_1x - b_1), \qquad (5)$$

$$y - m_1x - b_1 = k_3(y - m_2x - b_2). \qquad (6)$$

Now if these lines are to be AP', BP'' &c. of the figure, we must have their intercepts on Y equal respectively to $\frac{-c^2}{b_1}$, $\frac{-c^2}{b_2}$ and $\frac{c^2}{b_3}$; thus from (4) we shall have

$$\frac{b_2 - k_1b_3}{1 - k_1} = \frac{-c^2}{b_1}, \text{ whence } k_1 = \frac{c^2 + b_1b_2}{c^2 + b_3b_1}.$$

To obtain the lines AQ', BQ'' &c., we should have to make the intercepts equal to $\frac{c^2}{b_1}$, $\frac{c^2}{b_2}$ and $\frac{c^2}{b_3}$ which will give $k_1 = \frac{c^2 - b_1b_2}{c^2 - b_1b_3}$. If we write

$$k_1 = \frac{\pm c^2 - b_1 b_2}{\pm c^2 - b_3 b_1}$$

we shall cover both cases at once. Substituting this value of k_1 and values of k_2 and k_3 similarly obtained, in (4), (5) and (6), we have

$$[\pm c^2(m_1 - m_2) + b_3(m_2 b_1 - m_1 b_2)]x - b_3(b_1 - b_2)y \pm c^2(b_1 - b_2) = 0, \quad (7)$$

$$[\pm c^2(m_2 - m_3) + b_1(m_3 b_2 - m_2 b_3)]x - b_1(b_2 - b_3)y \pm c^2(b_2 - b_3) = 0, \quad (8)$$

$$[\pm c^2(m_3 - m_1) + b_2(m_1 b_3 - m_3 b_1)]x - b_2(b_3 - b_1)y \pm c^2(b_3 - b_1) = 0. \quad (9)$$

If (7), (8) and (9) be added they vanish identically, showing that the three lines represented by them pass through a single point. Thus the proposition is proved.

The following is a special case of this proposition.

Let the triangle ABC be inscribed in a conic: take any point P as a pole, and find its polar, say L: from the points P_1, P_2, P_3, where the sides of ABC cut L, draw lines through P, and find their poles, which will of course fall on L: call these poles P', P'' and P''': then $P'A$, $P''B$ and $P'''C$ meet in a point.

To prove this, draw the diameter through P, cutting L in O; then we have only to show that $OP_1 \times OP' = OP_2 \times OP'' = OP_3 \times OP'''$ = a constant.

L is parallel to the diameter conjugate to OP; referring the conic to this pair of conjugate diameters, let its equation be

$$\frac{x^2}{a_1^2} + \frac{y^2}{\pm b_1^2} = 1; \quad (10)$$

then that of its polar for the point (x_1, y_1) is

$$\frac{xx_1}{a_1^2} + \frac{yy_1}{\pm b_1^2} = 1. \quad (11)$$

For the point P, $y_1 = 0$ and $x_1 = a$, say; then

$$x = \frac{a_1^2}{a}. \quad (12)$$

Take any line through P, as

$$y = m(x - a). \quad (13)$$

To find the pole of this line compare (13) with (11), the conditions that they shall represent the same line are

$$m = -\frac{\pm b_1^2 x_1}{a_1^2 y_1} \text{ and } -ma = \frac{\pm b_1^2}{y_1}.$$

These conditions give

$$x_1 = \frac{a_1^2}{a}, \qquad y_1 = \frac{\pm b_1^2}{-ma}.$$

Now let us find the portion of L intercepted between the line of (13) and OP; $\therefore$ in (13) let $x = a_1^2 \div a$, and call the intercept y_2, therefore

$$y_2 = m\left(\frac{a_1^2}{a} - a\right) = m.\frac{a_1^2 - a^2}{a}.$$

Multiply together the values of y_1 and y_2, therefore

$$y_1 y_2 = -\frac{\pm b_1^2}{a^2}\left(a_1^2 - a^2\right), \text{ a constant.}$$

Hence $OP_1 \times OP' = OP_2 \times OP'' = OP_3 \times OP''' =$ a constant. This proves the proposition for the hyperbola and ellipse, and it may likewise be easily proved for the parabola. If P be a focus, of course L is a directrix.

If we eliminate c between any two of equations (7), (8), (9), we shall find the equation of the locus of O' and O'' when c varies. Solving for $\pm c^2$, and placing for brevity $b_1 - b_2 = \beta_3$, $b_2 - b_3 = \beta_1$, $b_3 - b_1 = \beta_2$, $m_1 - m_2 = \mu_3$, &c., and $m_2 b_1 - m_1 b_2 = \delta_3$ &c., we have

$$\pm c^2 = \frac{b_3(\beta_3 y - \delta_3 x)}{\beta_3 + \mu_3 x} = \frac{b_1(\beta_1 y - \delta_1 x)}{\beta_1 + \mu_1 x} = \frac{b_2(\beta_2 y - \delta_2 x)}{\beta_2 + \mu_2 x}.$$

Taking either pair of these values of $\pm c^2$ we shall arrive at the same result after expansion and collection of terms, viz.;

$$(m_1 b_1 \delta_1 + m_2 b_2 \delta_2 + m_3 b_3 \delta_3)x^2 - (\mu_1 b_2 b_3 + \mu_2 b_3 b_1 + \mu_3 b_1 b_2)xy$$
$$+ (b_1^2 \delta_1 + b_2^2 \delta_2 + b_3^2 \delta_3)x + \beta_1 \beta_2 \beta_3 y = 0. \quad (14)$$

This equation is perfectly symmetrical with respect to the three sides of the triangle, as it should be; and it represents a hyperbola. If we substitute A, B, C and D for the coefficients, so that the equation becomes

$$Ax^2 + Bxy + Cx + Dy = 0,$$

we find for the equations of the asymptotes

$$x = -\frac{D}{B}, \qquad y = -\frac{A}{B}x + \frac{AD - BC}{B^2}:$$

hence the coordinates of the center are

$$x = -\frac{D}{B} \text{ and } y = \frac{2AD - BC}{B^2}.$$

If the co-ordinates of the vertices of the triangle ABC which are respectively

$$x_1 = -\frac{\beta_1}{\mu_1},\ y_1 = -\frac{\delta_1}{\mu_1},\ x_2 = -\frac{\beta_2}{\mu_2},\ y_2 = -\frac{\delta_2}{\mu_2}, \text{ etc.,}$$

be substituted in the equation of the hyperbola, they will each be found to satisfy it; hence the hyperbola passes through the vertices of the triangle.